The Structure of Detour Degrees

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Abstract. We introduce a new subrecursive degree structure: *the structure of detour degrees*. We provide definition and basic properties of the detour degrees, including that they form a lattice and admit a jump operator. Our degree structure sheds light upon the open problems involving the small Grzegorcyk classes. There are also connections to complexity theory and complexity classes defined by resource-bounded Turing machines.

1 Introduction

Two main categories of subrecursive degrees are known from the literature: (i) degrees of honest functions and (ii) degrees of computable sets.

An honest function is a total computable and monotone increasing function with a simple graph, i.e., the characteristic function for the relation $f(x) = y$ is of degree $\mathbf{0}$. (E.g. if we are working with elementary degrees, the graph should be elementary; if we are working with primitive recursive degrees, the graph should be primitive recursive.) Typical reducibility relations between honest functions will be "being (Kalmar) elementary in" and "being primitive recursive in". Strong resource bounded reducibilities like "being polynomial time computable in" do not work when we are dealing with honest functions, and hence, computer scientists and complexity theoreticians have never been very attracted to honest degree theory. Honest subrecursive degree structures were studied in the early seventies by e.g. Basu [2], Meyer & Ritchie [20], Machtey [16][17][18]; and more recently by the first author, see [7][8][9][10]. The structure of honest elementary degrees is a distributive lattice with strong density properties. Kristiansen studies a jump operator on the honest elementary degrees, see e.g. [7] [10].

The study of degrees of computable sets started off in the mid seventies with Ladner's seminal paper [15]. In contrast to honest functions, computable sets admit degree structures induced by strong resource bounded reducibility relations, like e.g. "polynomial time computable in" and "logarithmic space computable in". The resulting degree structures, which of course are highly interesting from a complexity-theoretic point of view, are all upper semi lattices with strong density properties. See Merkle [19] for very general embedding results. Ladner [15] proved that neither the structure of polynomial time m-degrees nor the structure

of polynomial time T-degrees are lattices, and his proof methods and results generalise to wide variety of reducibilities. See Ambos-Spies [1] for more on degrees of computable sets.

In this paper we introduce a new subrecursive degree structure that does not fall into either of the two categories discussed above. The theory we develop has a flavour of honest degree theory, but our reducibility relation is sufficiently strong to make the theory interesting from a complexity-theoretic point of view. We prove that our degree structure is a distributive lattice, and we introduce a jump operator.

2 Preliminaries

We will use some notation and terminology from Clote [3]. An *operator*, here also called *(definition) scheme*, is a mapping from functions to functions. Let $\mathcal{X}$ be a set of functions (possibly given in a slightly informal notation), and let OP be a collection of operators. The *function algebra* $[\mathcal{X}; \text{OP}]$ is the smallest set of functions containing $\mathcal{X}$ and closed under the operations of OP. Let COMP denote the scheme of *composition*, i.e. the scheme $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$ where $m \geq 0$, and let PR denotes the scheme of *primitive recursion*, that is, the scheme

$$f(\vec{x}, 0) = g(\vec{x}) \qquad f(\vec{x}, y+1) = h(\vec{x}, y, f(\vec{x}, y)) \; .$$

Further, *bounded minimalisation* is the scheme $f(\vec{x}) = (\mu i \leq y)[R(\vec{x}, i)]$ where

$$(\mu i \leq y)[R(\vec{x}, i)] = \begin{cases} \text{the least } i \leq y \text{ such that the relation } R(\vec{x}, i) \text{ holds} \\ 0 \text{ if no such } i \text{ exists.} \end{cases}$$

Any function $f : \mathbb{N}^n \to \mathbb{N}$ will be identified with a relation $R_f \subseteq \mathbb{N}^n$. The relation $R_f(\vec{n})$ holds iff $f(\vec{x}) = 0$, and the relation R_f belongs to a class of functions $\mathcal{F}$ iff f belongs to $\mathcal{F}$. A *problem* is a subset of $\mathbb{N}$, or equivalently, a unary relation. We will use $A, B, C, \ldots$ to denote problems, and a function f *decides* a problem A when $f(x) = 0$ iff $x \in A$. For any set $\mathcal{F}$ of number-theoretic functions, $\mathcal{F}_*$ denotes the set of problem decided by the functions in $\mathcal{F}$.[4]

Let $I_i^n : \mathbb{N}^n \to \mathbb{N}$ denote the projection function, i.e. $I_i^n(x_1, \ldots, x_n) = x_i$ where $\vec{x} = x_1, \ldots, x_n$ and $1 \leq i \leq n$. Let I denote the set of all such projection functions. Let C_k denote the constant function yielding the natural number k. The *small Grzegorczyk classes* are defined by $\mathcal{E}^0 := [I, C_0, S; \text{COMP}, \text{BR}]$, $\mathcal{E}^1 := [I, C_0, S, +; \text{COMP}, \text{BR}]$ and $\mathcal{E}^2 := [I, C_0, S, +, x^2 + 2; \text{COMP}, \text{BR}]$ where S denotes the successor function and BR denotes the scheme of bounded primitive recursion, that is, the scheme

$$f(\vec{x}, 0) = g(\vec{x}) \qquad f(\vec{x}, y+1) = h(\vec{x}, y, f(\vec{x}, y)) \qquad f(\vec{x}, y) \leq j(\vec{x}, y) \; .$$

[4] Our use of the subscript $*$ differs slightly from the standard in the literature. Normally, $\mathcal{F}_*$ denotes the 0-1 valued functions in $\mathcal{F}$ whereas we use $\mathcal{F}_*$ to denote the set of problem decided by the functions in $\mathcal{F}$. This is a matter of convenience, and the deviation has no essential mathematical implications.

3 Detour Functions and $\mathcal{P}^{\dot{-}}$-computability

Definition. $\mathcal{P}^{\dot{-}} := [I, C_1; \text{COMP}, \text{PR}]$. □

Roughly speaking, $\mathcal{P}^{\dot{-}}$ contains the primitive recursive functions which can be defined without the successor function, and it is easy to see that we have $f(\vec{x}) \leq \max(\vec{x}, 1)$ for any $f \in \mathcal{P}^{\dot{-}}$. Even though $\mathcal{P}^{\dot{-}}$ contains only non-increasing functions, surprisingly powerful relations and predicates belong to the class. Indeed, it is an open problem if $\mathcal{P}^{\dot{-}}_* = \text{LINSPACE}$; we have $\mathcal{P}^{\dot{-}}_* = \mathcal{E}^0_* \subseteq \mathcal{E}^1_* \subseteq \mathcal{E}^2_* = \text{LINSPACE}$, and it is not known if any of the inclusions are strict.[5] For more on $\mathcal{P}^{\dot{-}}$ and similar non-increasing function algebras, see e.g. Kristiansen [5] [6] and Kristiansen & Voda [13] [14].

Lemma 1 (Basic functions). *The following number-theoretic functions belong to $\mathcal{P}^{\dot{-}}$: (i) 0,1 (constant functions); (ii) P(x) (predecessor); (iii) $x \dot{-} y$ (modified subtraction); (iv) c where $c(x, y_1, y_2) = y_1$ if $x = 0$ and $c(x, y_1, y_2) = y_2$ if $x \neq 0$; (v) $\max(x, y)$ and $\min(x, y)$; (vi) f where $f(x, m) = x + 1 \pmod{m+1}$ for $x \leq m$; (vii) f where $f(x, y, m) = x + y \pmod{m+1}$ for $x, y \leq m$; (viii) f where $f(x, y, m) = x \times y \pmod{m+1}$ for $x, y \leq m$; (ix) f where $f(x, y, m) = x^y \pmod{m+1}$ for $x, y \leq m$; (x) for each $k \in \mathbb{N}$, the almost everywhere constant function ν_k where $\nu_k(x) = \min(x, k)$. Moreover, the class $\mathcal{P}^{\dot{-}}$ is closed under bounded minimalisation.*

Proof. To define the constant function 0 is slightly nontrivial. Define g by primitive recursion such that $g(x, 0) = x$ and $g(x, y+1) = y$. Then we can define the predecessor P from g since $P(x) = g(x, x)$. Further, we can define the constant function 0 by $0 = P(1)$. This proves that (i) and (ii) hold. (iii) holds since we have $x \dot{-} 0 = x$ and $x \dot{-} (y+1) = P(x \dot{-} y)$. (iv) holds since $c(0, y_1, y_2) = I^2_1(y_1, y_2)$ and $c(x+1, y_1, y_2) = I^4_2(y_1, y_2, x, c(x, y_1, y_2))$. Furthermore, (v) holds since $\max(x, y) = c(1 \dot{-} (x \dot{-} y), x, y)$ and $\min(x, y) = c(1 \dot{-} (x \dot{-} y), y, x)$. (vi) holds since $c(m \dot{-} x, 0, m \dot{-} ((m \dot{-} x) \dot{-} 1)) = x + 1 \pmod{m+1}$ for $x \leq m$. We omit the rest of the proof. □

Lemma 2 (Basic relations). *(i) The relations $x \leq y$ and $x = y$ belong to $\mathcal{P}^{\dot{-}}$. (ii) The relations of $\mathcal{P}^{\dot{-}}$ are closed under the propositional operations (and, or, not) and under bounded existential and bounded universal quantification, i.e., $\exists x \leq y[\ldots]$ and $\forall x \leq y[\ldots]$.*

Proof. Apply the functions given by Lemma 1. We have $y \dot{-} x = 0$ iff $x \leq y$; we have $c(f(\vec{x}), c(g(\vec{x}), 0, 1), 1) = 0$ iff $f(\vec{x}) = 0$ and $g(\vec{x}) = 0$; we have $c(f(\vec{x}), 1, 0) = 0$ iff $f(\vec{x}) \neq 0$; and so on. Use the bounded minimalisation operator to define the bounded quantifiers. □

[5] The notorious problem $\mathcal{E}^0_* \overset{?}{\subseteq} \mathcal{E}^1_* \overset{?}{\subseteq} \mathcal{E}^2_*$ is more than 50 years old and was posed in Grzegorczyk [4]. It is trivial that $\mathcal{P}^{\dot{-}}_* \subseteq \mathcal{E}^0_*$, and it follows from Theorem 5 in this paper that $\mathcal{P}^{\dot{-}}_* = \mathcal{E}^0_*$. (The problem $\mathcal{P}^{\dot{-}}_* \overset{?}{\subseteq} \mathcal{E}^0_*$ was posed as open in Kristiansen & Barra [11].) Ritchie [22] proved that $\mathcal{E}^2_* = \text{LINSPACE}$.

Definition. A *detour function* $f : \mathbb{N} \to \mathbb{N}$ satisfies the following requirements: (1) $x \leq f(x)$; (2) $f(x) \leq f(x+1)$; and (3) the graph of f, i.e. the relation $f(x) = y$, is in $\mathcal{P}^-$. Henceforth we will use Latin letters f, g, h, etc., to denote detour functions. and Greek letters ϕ, ψ, ξ, etc., to denote arbitrary functions. A function ψ is *$\mathcal{P}^-$-computable in the detour function f* if there exists $\phi \in \mathcal{P}^-$ such that $\psi(\vec{x}) = \phi(\vec{x}, z)$ for all $z \geq f(\max(\vec{x}))$. Let $\mathcal{P}^-(f)$ denote the set of functions $\mathcal{P}^-$-computable in f. □

What motivates our definitions? Well, we are mainly interested in $\mathcal{P}^-(f)_*$, that is, the set of problems $\mathcal{P}^-$-computable in f. A problem A will be $\mathcal{P}^-$-computable in a detour function f if f provides enough resources for a $\mathcal{P}^-$-computation to solve the problem. Any $\mathcal{P}^-$-function is non-increasing, and thus, a $\mathcal{P}^-$-computation solving a problem on input x in a detour function f, can make *detours* via any number less than $f(x)$, but no further detours are possible. Our definitions assure that if g provides more resources (longer detours) than f, that is if $f(x) \leq g(x)$, then any problem $\mathcal{P}^-$-computable in f will also be $\mathcal{P}^-$-computable in g.

It is essential that we require the graph of a detour function to be in $\mathcal{P}^-$. This prevents us from coding a set A of high computational complexity into a detour function f, e.g. by letting $f(x) = 2x$ when $x \in A$ and $f(x) = 2x+1$ when $x \notin A$, and thus, the requirement ensures that a detour function provides nothing but resources to a $\mathcal{P}^-$-computation. (Without the requirement our degree structure would simply degenerate to a degree structure of computable sets.)

Definition. For any set $A \subseteq \mathbb{N}$, we define the *characteristic function* χ_A by $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = 1$ if $x \notin A$. We define the function f^{-1} by $f^{-1}(x) = (\mu i \leq x)[f(i) \geq x]$, and when f is a detour function, we will say that f^{-1} is the *inverse function* of f. For any $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, we define the functions $\max[f, g]$ and $\min[f, g]$ by respectively $\max[f, g](x) = \max(f(x), g(x))$ and $\min[f, g](x) = \min(f(x), g(x))$ □

The two following lemmas are collections of very basic facts. The lemmas will be applied frequently throughout the paper, but we will only occasionally refer explicitly to the lemmas.

Lemma 3. *Let f be any detour function. (i) There exists $\xi \in \mathcal{P}^-$ such that $\phi(\vec{x}) = \xi(\vec{x}, f(\max(\vec{x})))$ if, and only if, there exist $\psi \in \mathcal{P}^-$ such that $\phi(\vec{x}) = \psi(\vec{x}, z)$ for any $z \geq f(\max(\vec{x}))$; (ii) $A \in \mathcal{P}^-(f)_*$ iff $\chi_A \in \mathcal{P}^-(f)$; (iii) $\mathcal{P}^-(f)_*$ is closed under unions, intersections and complements; (iv) $f^{-1} \in \mathcal{P}^-$ and $f^{-1}(f(x)) \leq x$, besides, if f is strictly monotone, we also have $f^{-1}(f(x)) = x$.*

Proof. We prove the only-if-direction of (i). (The if-direction is trivial.) Assume there exists $\xi \in \mathcal{P}^-$ such that $\phi(\vec{x}) = \xi(\vec{x}, f(\max(\vec{x})))$. Let $\xi_0(\vec{x}, y) = (\mu i \leq y)[f(\max(\vec{x})) = i]$. The graph of f belongs to $\mathcal{P}^-$, and $\mathcal{P}^-$ is closed under bounded minimalisation, and thus, we have $\xi_0 \in \mathcal{P}^-$. Let $\xi_1(\vec{x}, z) = \xi(\vec{x}, \xi_0(\vec{x}, z))$. Now we have $\xi_1 \in \mathcal{P}^-$ and $\phi(\vec{x}) = \xi_1(\vec{x}, z)$ for any $z \geq f(\max(\vec{x}))$. Next we prove (iv). By the definition of f^{-1} we have $f^{-1}(f(x)) = (\mu i \leq f(x))[f(i) \geq f(x)]$. Since f is monotone, it cannot be the case that $f^{-1}(f(x)) > x$, and if f is strictly

monotone, it cannot be the case that $f^{-1}(f(x)) < x$. Thus, $f^{-1}(f(x)) = x$ when f is strictly monotone. The graph of f belongs to $\mathcal{P}^-$, and $\mathcal{P}^-$ is closed under bounded minimalisation. Thus, $f^{-1} \in \mathcal{P}^-$. We omit the rest of the proof. □

Lemma 4. *(i) Any non-constant polynomial $p(x)$ (generated from x and the constants of $\mathbb{N}$ by $+$ and $\times$) is a detour function. (ii) The function 2^x is a detour function. (iii) The detour functions are closed under compositions. (iv)* $\max[f,g]$ *and* $\min[f,g]$ *are detour functions if f and g are detour functions.*

Proof. (i) and (ii) follow from Lemma 1. Let f and g be detour functions. Obviously, we have $x \leq f(g(x))$ and $f(g(x)) \leq f(g(x+1))$, and thus, (iii) follows from Lemma 2 since $f(g(x)) = y \;\Leftrightarrow\; \exists z \leq y[g(x) = z \;\wedge\; f(z) = y]$. Further, (iv) follows from Lemma 2 since $\max[f,g](x) = y \;\Leftrightarrow$

$$(f(x) = y \;\wedge\; \exists z \leq y[g(x) = z]) \;\vee\; (g(x) = y \;\wedge\; \exists z \leq y[f(x) = z])$$

and $\min[f,g](x) = y \;\Leftrightarrow$

$$(f(x) = y \;\wedge\; \neg\exists z < y[g(x) = z]) \;\vee\; (g(x) = y \;\wedge\; \neg\exists z < y[f(x) = z])\,.$$

□

4 The Lattice of Detour Degrees

Definition. We define $f \preceq g :\Leftrightarrow \mathcal{P}^-(f)_* \subseteq \mathcal{P}^-(g)_*$; $f \equiv g :\Leftrightarrow f \preceq g \wedge g \preceq f$; $f \prec g :\Leftrightarrow f \preceq g \wedge g \not\preceq f$. Now, $\equiv$ is an equivalence relation, and the *detour degrees* are the $\equiv$-equivalence classes of the detour functions. We denote the equivalence class of f by $\mathrm{dg}(f)$, that is, $\mathrm{dg}(f) = \{g \mid g \equiv f\}$, and following the tradition of classical computability theory, we use boldface lowercase Latin letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ to denote our degrees. Furthermore, we use $<$ and $\leq$ to denote the relations induced on the degrees by respectively $\prec$ and $\preceq$, and we use $\mathcal{D}$ to denote the set of detour degrees. □

Lemma 5. *Let f, g, h be detour functions. (i) If $h \preceq f$ and $h \preceq g$, then $h \preceq$* $\min[f,g]$*. (ii) If $f \preceq h$ and $g \preceq h$, then* $\max[f,g] \preceq h$.

Proof. The following simple claim is the key to the proof.

> **(Claim)** Let f and g be detour functions, and let $B = \{x | f(x) \leq g(x)\}$. We have $B \in \mathcal{P}^-(\min[f,g])_*$ (and thus $B \in \mathcal{P}^-(f)_*$ and $B \in \mathcal{P}^-(g)_*$).

Let

$$\xi(x) \;=\; (\mu i \leq \min[f,g](x))\,[f(x) = i \vee g(x) = i]$$

and let $\chi_B(x) = c(f(x) = \xi(x), 0, 1)$. Now, χ_B is the characteristic function of B, and $\chi_B \in \mathcal{P}^-(\min[f,g])$ by the lemmas in Section 3. This proves the claim.

We prove (i). Assume $h \preceq f$ and $h \preceq g$ (*). (We will prove $h \preceq \min[f,g]$.) Assume $A \in \mathcal{P}^-(h)_*$. (We will prove $A \in \mathcal{P}^-(\min[f,g])_*$.) By (*), we have

$A \in \mathcal{P}^{-}(f)_*$ and $A \in \mathcal{P}^{-}(g)_*$, and hence, we have $\phi_1, \phi_2 \in \mathcal{P}^{-}$ such that $\chi_A(x) = \phi_1(x, f(x))$ and $\chi_A(x) = \phi_2(x, g(x))$. By (Claim) we have $\eta \in \mathcal{P}^{-}$ such that $\eta(x, \min[f,g](x)) = 0$ iff $f(x) \leq g(x)$. Let $\psi(x,z) = c(\eta(x,z), \phi_1(x,z), \phi_2(x,z))$. Then, $\psi \in \mathcal{P}^{-}$ and $\chi_A(x) = \psi(x, \min[f,g](x))$, and thus, $A \in \mathcal{P}^{-}(\max[f,g])_*$. This completes the proof of (i).

In order to prove (ii), we assume $f \preceq h$ and $g \preceq h$ (*) and prove $\max[f,g] \preceq h$. To prove $\max[f,g] \preceq h$, we assume $A \in \mathcal{P}^{-}(\max[f,g])_*$ and prove $A \in \mathcal{P}^{-}(h)_*$. Since $A \in \mathcal{P}^{-}(\max[f,g])_*$, we have $\phi \in \mathcal{P}^{-}$ such that $\chi_A(x) = \phi(x, \max[f,g](x))$. Let $B = \{x | f(x) \leq g(x)\}$, and let

$$x \in A_1 \;\Leftrightarrow\; \phi(x, f(x)) = 0 \wedge x \notin B \quad \text{and} \quad x \in A_2 \;\Leftrightarrow\; \phi(x, g(x)) = 0 \wedge x \in B\,.$$

By (Claim), we have $A_1 \in \mathcal{P}^{-}(f)_*$ and $A_2 \in \mathcal{P}^{-}(g)_*$, and by (*), we have $A_1, A_2 \in \mathcal{P}^{-}(h)_*$. By Lemma 3, we have $A_1 \cup A_2 \in \mathcal{P}^{-}(h)_*$. This completes the proof because $A = A_1 \cup A_2$. □

Lemma 6. *For any detour functions f, f_1, g, g_1, we have (i) if $f \equiv f_1$ and $g \equiv g_1$, then $\max[f,g] \equiv \max[f_1, g_1]$; (ii) if $f \equiv f_1$ and $g \equiv g_1$, then $\min[f,g] \equiv \min[f_1, g_1]$.*

Proof. We prove $f \preceq f_1 \wedge g \preceq g_1 \Rightarrow \max[f,g] \preceq \max[f_1, g_1]$. It follows that (i) holds. Assume $f \preceq f_1$ and $g \preceq g_1$. Since $f_1 \preceq \max[f_1, g_1]$ and $g_1 \preceq \max[f_1, g_1]$, we have $f \preceq \max[f_1, g_1]$ and $g \preceq \max[f_1, g_1]$. By Lemma 5 (ii), we have $\max[f,g] \preceq \max[f_1, g_1]$. This completes the proof of (i). The proof of (ii) is symmetric; apply Lemma 5 (i) in place of Lemma 5 (ii). □

Lemma 4 (iv) and Lemma 6 show that the minimum and the maximum operators on the detour functions induce operators on the detour degrees, and thus, the next definition make sense.

Definition. Let $f \in \mathbf{a}$ and $g \in \mathbf{b}$ be detour functions. We define $\mathbf{a} \cup \mathbf{b}$ and $\mathbf{a} \cap \mathbf{b}$ by respectively $\mathrm{dg}(\max[f,g])$ and $\mathrm{dg}(\min[f,g])$. □

Theorem 1 (The Lattice of Detour Degrees). *The structure $\langle \mathcal{D}, \leq, \cup, \cap \rangle$ is a distributive lattice, that is, for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{D}$ we have (i) $\mathbf{a} \cap \mathbf{b}$ is the greatest lower bound (glb) of $\mathbf{a}$ and $\mathbf{b}$ under the ordering $\leq$; (ii) $\mathbf{a} \cup \mathbf{b}$ is the least upper bound (lub) of $\mathbf{a}$ and $\mathbf{b}$ under the ordering $\leq$; (iii) $\mathbf{a} \cup (\mathbf{b} \cap \mathbf{c}) = (\mathbf{a} \cup \mathbf{b}) \cap (\mathbf{a} \cup \mathbf{c})$ and $\mathbf{a} \cap (\mathbf{b} \cup \mathbf{c}) = (\mathbf{a} \cap \mathbf{b}) \cup (\mathbf{a} \cap \mathbf{c})$.*

Proof. We prove (i). Since we have $\min[f,g] \preceq f$ and $\min[f,g] \preceq g$, it is obvious that $\mathbf{a} \cap \mathbf{b}$ is a lower bound of $\mathbf{a}$ and $\mathbf{b}$. To see that $\mathbf{a} \cap \mathbf{b}$ indeed is the greatest lower bound of $\mathbf{a}$ and $\mathbf{b}$, pick any degree $\mathbf{c}$ that lies below both $\mathbf{a}$ and $\mathbf{b}$. By Lemma 5 (i), we have that $\mathbf{c}$ also lies below $\mathbf{a} \cap \mathbf{b}$. Thus, (i) holds. The proof of (ii) is symmetric (use Lemma 5 (ii) in place of Lemma 5 (i)). Finally, (iii) holds since the max operator distributes over the min operator and vice versa. □

5 The Jump Operator

Lemma 7. *Let f, g, h be detour functions, and let h be strictly monotone. If $f \preceq g$, then $f \circ h \preceq g \circ h$.*

Proof. For any set A, let $\#_A(x) = \sum_{i \leq x} 1 \dot{-} \chi_A(i)$. The function $\#_A(x)$ gives the cardinality of the set $\{y \mid y \in A \wedge y \leq x\}$, and when $f(x) \geq x + 1$, we have

$$A \in \mathcal{P}^{-}(f)_* \;\Leftrightarrow\; \#_A \in \mathcal{P}^{-}(f) \qquad \text{(Claim)}$$

We skip the proof of the claim and assume $f \preceq g$. (We will prove $f \circ h \preceq g \circ h$.)

To prove $f \circ h \preceq g \circ h$, we have to prove that $A \in \mathcal{P}^{-}(f \circ h)_*$ entails $A \in \mathcal{P}^{-}(g \circ h)_*$. We assume $A \in \mathcal{P}^{-}(f \circ h)_*$ (and prove $A \in \mathcal{P}^{-}(g \circ h)_*$.)

By (Claim), we have $\#_A \in \mathcal{P}^{-}(f \circ h)$, and thus, there exists $\phi \in \mathcal{P}^{-}$ such that $\#_A(x) = \phi(x, fh(x))$. Let B be the set given by $\#_B(x) = \phi(h^{-1}(x), f(x))$. Then, $B \in \mathcal{P}^{-}(f)_*$ and

$$\#_B(h(x)) \;=\; \phi(h^{-1}h(x), fh(x)) \;=\; \phi(x, fh(x)) \;=\; \#_A(x)\,.$$

Now, since $B \in \mathcal{P}^{-}(f)_*$ and $f \preceq g$, we have $B \in \mathcal{P}^{-}(g)_*$, and then, by (Claim), there exists $\psi \in \mathcal{P}^{-}$ such that $\#_B(x) = \psi(x, g(x))$. Thus,

$$\#_A(x) \;=\; \#_B(h(x)) \;=\; \psi(h(x), gh(x))\,. \qquad (*)$$

Let $h'(x, y) = (\mu z \leq y)[h(x) = z]$ and $\xi(x, y) = \psi(h'(x, y), y)$. Then, $\xi \in \mathcal{P}^{-}$ and

$$\xi(x, gh(x)) \;=\; \psi(h'(x, gh(x)), gh(x)) \;=\; \psi(h(x), gh(x)) \;\overset{(*)}{=}\; \#_A(x)\,.$$

Thus, we have $\#_A \in \mathcal{P}^{-}(g \circ h)$, and by (Claim), we also have $A \in \mathcal{P}^{-}(g \circ h)_*$. □

It follows from Lemma 7 that $f \equiv g \;\Rightarrow\; f \circ h \equiv g \circ h$ for any detour functions f, g and any strictly monotone detour function h. Now, 2^x is a strictly monotone detour function, and furthermore, the detour functions are closed under composition. Hence, the next definition makes sense.

Definition. We define the *jump operator* $\cdot'$ on the detour functions by $f'(x) = f(2^x)$. We define the *jump operator* $\cdot'$ on the detour degrees by $\mathbf{a}' = \mathrm{dg}(f')$ where f is some detour function such that $\mathbf{a} = \mathrm{dg}(f)$. Let $\mathbf{0}$ denote the degree $\mathrm{dg}(I_1^1)$. □

It is easy to see that we have $\mathbf{0} \leq \mathbf{a}$ for any degree $\mathbf{a}$. (We have $\mathbf{a} = \mathrm{dg}(f)$ for some f such that $f(x) \geq I_1^1(x) = x$, and thus $\mathcal{P}^{-}(I_1^1)_* \subseteq \mathcal{P}^{-}(f)_*$.) To prove that $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \mathbf{0}''' < \ldots$, we have to resort to a machine model of computation. We have chosen the Turing machines.

Lemma 8. *There exists $\phi \in \mathcal{P}^{-}$ such that for any $A \in \mathcal{P}^{-}(f)_*$ we have a fixed $k \in \mathbb{N}$ and fixed $e_0, e_1, e_2, \ldots \in \mathbb{N}$ such that*

$$\chi_A(x) = \phi(e_i, x, z) \quad \textit{for any } z \geq f(x)^k \textit{ and any } i \in \mathbb{N}.$$

Proof. We will work with Turing machines accepting pairs of natural numbers. The two inputs numbers are given to the machines in dyadic notation, and $|z|$ denotes the number of bits required to represent the number z. We assume some computable enumeration $M_0, M_1, M_2, \ldots$ of such machines, furthermore, we can w.l.o.g. assume that each machine occurs infinitely many times in the enumeration. We define the function Φ: Let $\Phi(e,x,y,z) = 0$ if M_e accepts the input x, y in space $|z|$ and time z; let $\Phi(e,x,y,z) = 1$ otherwise. We skip the proof of the following claim.

(Claim 1) $\Phi \in \mathcal{P}^{-}$.

Obviously, we have $\mathcal{P}^{-} \subseteq \mathcal{E}^2$, and it is well known that any function in $\mathcal{E}^2$ can be computed by a Turing machine working in linear space in the length of the input (Ritchie [22]). Thus, if ϕ is a binary function in $\mathcal{P}^{-}$, then $\phi(x,y)$ is computable by a Turing machine M running in space $k_1|\max(x,y)|$ for some fixed $k_1 \in \mathbb{N}$. Now, if M runs in space $k_1|\max(x,y)|$, there also exists a fixed $k \in \mathbb{N}$ such that M runs in space $|\max(x,y)^k|$ and time $\max(x,y)^k$. Hence, the next claim holds.

(Claim 2) Let ψ be a binary 0-1 valued function in $\mathcal{P}^{-}$. There exists a fixed $k \in \mathbb{N}$ and infinitely many $e \in \mathbb{N}$ such that $\psi(x,y) = \Phi(e,x,y,\max(x,y)^k)$.

Let $\langle \cdot,\cdot \rangle$ be a standard bijection from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$, e.g. $\langle x,y \rangle = (\sum_{i \leq x+y} i) + y$, and let π_1, π_2 be the corresponding decoding functions, i.e. $\pi_1(\langle x,y \rangle) = x$ and $\pi_2(\langle x,y \rangle) = y$. The pairing function is not in $\mathcal{P}^{-}$, but the decoding functions π_1, π_2 are. Furthermore, the binary function $\lfloor x^{\frac{1}{y}} \rfloor$ belongs to $\mathcal{P}^{-}$. We define the function ϕ by

$$\phi(w,x,y) \;=\; \Phi(\pi_1(w), x, \lfloor y^{\frac{1}{\pi_2(w)}} \rfloor, y)\,.$$

It follows from our discussion and (Claim 1) that we have $\phi \in \mathcal{P}^{-}$. To complete our proof, we will prove that for any $A \in \mathcal{P}^{-}(f)_*$ there exist $k \in \mathbb{N}$ and infinitely many $e \in \mathbb{N}$ such that $\chi_A(x) = \phi(e,x,z)$ holds for any $z \geq f(x)^k$.

By the definition of ϕ we have

$$\phi(\langle e,k \rangle, x, z^k) \;=\; \Phi(e,x,z,z^k)\,. \qquad (*)$$

Let A be any set in $\mathcal{P}^{-}(f)_*$. Then we have $\psi \in \mathcal{P}^{-}$ such that $\chi_A(x) = \psi(x,z)$ for any $z \geq f(x)$. Furthermore, for some e, k and and any $z \geq f(x)$, we have

$$\begin{aligned}
\chi_A(x) &= \psi(x,z) \\
&= \Phi(e,x,z,\max(x,z)^k) && \text{(Claim 2)} \\
&= \Phi(e,x,z,z^k) && \text{since } x \leq f(x) \leq z \\
&= \phi(\langle e,k \rangle, x, z^k)\,. && (*)
\end{aligned}$$

and indeed, by (Claim 2) there are infinitely many numbers e such that these equalities hold. Let $d_0, d_1, d_2, \ldots$ denote these numbers, and let $e_i = \langle d_i, k \rangle$. Now we have $\chi_A(x) = \phi(e_i, x, z^k)$ for any $i \in \mathbb{N}$ and any $z \geq f(x)$, and thus, $\chi_A(x) = \phi(e_i, x, z)$ for any $i \in \mathbb{N}$ and any $z \geq f(x)^k$. □

Theorem 2. *Let f be any detour function such that for any $k \in \mathbb{N}$ we have $f(x)^k \leq f(2^x)$ for all but finitely many x. Then, we have $f' \not\leq f$.*

Proof. Let B be the set given by

$$\chi_B(x) = \begin{cases} 0 \text{ if } \phi(x, x, f'(x)) = 1 \\ 1 \text{ otherwise.} \end{cases} \qquad (*)$$

where ϕ is the function given by Lemma 8. Obviously, we have $B \in \mathcal{P}^{-}(f')$. We prove that $B \notin \mathcal{P}^{-}(f)$.

Assume for the sake of a contradiction that $B \in \mathcal{P}^{-}(f)$. Then, by Lemma 8 and (*), we have e_i and k such that

$$\chi_B(e_i) \;=\; \phi(e_i, e_i, f(e_i)^k) \;=\; \phi(e_i, e_i, f'(e_i)) \;\neq\; \chi_B(e_i)\,.$$

(Since we have infinitely many e_i's, it is possible to pick e_i such that $f(e_i)^k \leq f(2^{e_i}) = f'(e_i)$.) It is a contradiction that $\chi_B(e_i) \neq \chi_B(e_i)$. □

Theorem 3. *Let $\mathbf{0}^0 = \mathbf{0}$ and $\mathbf{0}^{n+1} = (\mathbf{0}^n)'$. We have $\mathbf{0}^n < \mathbf{0}^{n+1}$ for any $n \in \mathbb{N}$.*

Proof. It is trivial that $\mathbf{0}^n \leq \mathbf{0}^{n+1}$. Let $g_0(x) = x$ and $g_{i+1}(x) = 2^{g_i(x)}$. Then we have $\mathrm{dg}(g_i) = \mathbf{0}^i$. Further, for any fixed k, we have $g_n(x)^k < g_n(2^x) = g_{n+1}(x)$ for all sufficiently large x. Hence, we have $\mathbf{0}^{n+1} \not\leq \mathbf{0}^n$ by Theorem 2. □

Theorem 4 (Jump Inversion). *Let $\mathbf{a}$ be any detour degree above $\mathbf{0}'$, i.e. $\mathbf{0}' \leq \mathbf{a}$. There exists a detour degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$.*

Proof. Let $\mathbf{a} = \mathrm{dg}(f)$. We can w.l.o.g. assume that $f(x) \geq 2^x$. Let $\lceil \log_2 x \rceil$ be the inverse function of 2^x, and let $g(x) = f(\lceil \log_2 x \rceil)$. Now, g is a detour function, and obviously we have $g'(x) = f(\lceil \log_2 2^x \rceil) = f(x)$, and hence, $\mathrm{dg}(g)' = \mathbf{a}$. □

6 A Few Theorems and Some Comments

The implication $f(x) \leq g(x) \Rightarrow \mathrm{dg}(f) \leq \mathrm{dg}(g)$ follows straightforwardly from our definitions. The reversed implication is not true. (So, our degree structure is not trivial.)

Theorem 5. $\mathcal{P}^{-}(x+1)_* \subseteq \mathcal{P}^{-}(x)_*$.

By Lemma 7 and Theorem 5, we have $\mathrm{dg}(f(x)+a) = \mathrm{dg}(f(x)+b)$ for any $a, b \in \mathbb{N}$ and any detour function f. The long and technical proof of the Theorem 5 is available in a preprint posted on the Internet [12]. By extending the ideas of this proof we expect to be able to prove that we have $\mathrm{dg}(x^n) = \mathrm{dg}(x^n + p(x))$ for any $n \in \mathbb{N}$ and any detour polynomial p of degree strictly less than n.

Theorem 6. *For any decidable problem (computable set) A there exists a detour function f such that $A \in \mathcal{P}^{-}(f)_*$.*

Proof. Let A be any computable set. By Kleene's Normal Form Theorem (see e.g. Odifreddi [21] p. 179) there exists a primitive recursive function $\mathcal{U}$ and a primitive recursive relation $\mathcal{T}$ such that $\chi_A(x) = \mathcal{U}((\mu i)[\mathcal{T}(x,i)])$. It can be proved that $\mathcal{U}, \mathcal{T} \in \mathcal{P}^{-}$. Let $g(x) = (\mu i)[\mathcal{T}(x,i)]$ and $f(x) = \sum_{i \leq x} g(i)$. Now, f is a detour function and $A \in \mathcal{P}^{-}(f)_*$. □

The previous theorem shows that the detour degrees span all the computable sets. The next theorems show that the degree structure is sufficiently fine-grained to shed light upon (open) problems in subrecursion theory and complexity theory. We omit the proofs of these theorems due to space limitations. (Recall that any unary number-theoretic function generated by composition from constants, $+$, $\times$ and 2^x, is a detour function. The small Grzegorczyk classes are defined in Section 2.)

Theorem 7 (Turing Machine Space Classes). *For $i \in \mathbb{N}$, let* SPACE 2_i^{LIN} *be the set of problems decidable by a deterministic Turing machine working in space $2_i^{c|x|}$ for some $c \in \mathbb{N}$ (where $2_0^x = x$ and $2_{i+1}^x = 2^{2_i^x}$). Then, $A \in$* SPACE 2_i^{LIN} *iff $A \in \mathcal{P}^{-}(2_i^{p(x)})_*$ for some polynomial p. Moreover, for any $i \in \mathbb{N}$ and any polynomial p, the function $2_i^{p(x)}$ is a detour function*

Theorem 8 (Small Grzegorczyk Classes). *(i) $A \in \mathcal{E}_*^0$ iff $A \in \mathcal{P}^{-}(x+k)_*$ for some k. (ii) $A \in \mathcal{E}_*^1$ iff $A \in \mathcal{P}^{-}(kx)_*$ for some k. (iii) $A \in \mathcal{E}_*^2$ iff $A \in \mathcal{P}^{-}(p)_*$ for some polynomial p.*

Now, it follows from Theorem 5 that $\mathcal{P}_*^{-} = \mathcal{E}_*^0$, and it is an open problem whether any of the inclusions $\mathcal{E}_*^1 \subseteq \mathcal{E}_*^2 \subseteq \mathcal{E}_*^2$ are strict. Thus, it will be very hard to prove that $\mathbf{0} \neq \mathrm{dg}(p)$ for some polynomial p. It might even not be true. However, in our degree structure the open problem $\mathcal{E}_*^0 \stackrel{?}{\subseteq} \mathcal{E}_*^1 \stackrel{?}{\subseteq} \mathcal{E}_*^2$ manifests itself as a special case of a more general open problem: Do there exist $n \in \mathbb{N}$ and a detour polynomial p such that $\mathbf{0}^n < \mathrm{dg}(2_n^{p(x)})$?

Finally, we will argue that there exist incomparable degrees below $\mathbf{0}'$, i.e., we have $\mathbf{a}, \mathbf{b} < \mathbf{0}'$ such that $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} \not\leq \mathbf{a}$. Let $d_0 = 0$ and $d_{i+1} = 2^{d_i}$, and for $j = 0, 1, 2$, let

$$f_j(x) = \begin{cases} 2^x & \text{if } d_{3i+j} \leq x < d_{3i+j+1} \\ \max(f_j(x \dot{-} 1), x) & \text{otherwise.} \end{cases}$$

Now, f_0, f_1, f_2 are detour functions such that $\max(f_0(x), f_1(x), f_2(x)) = 2^x$ and $\min(f_0(x), f_1(x), f_2(x)) = x$. Let $\mathbf{a} = \mathrm{dg}(f_0)$, $\mathbf{b} = \mathrm{dg}(f_1)$ and $\mathbf{c} = \mathrm{dg}(f_2)$. Then, we have $\mathbf{a} \cup \mathbf{b} \cup \mathbf{c} = \mathbf{0}'$ and $\mathbf{a} \cap \mathbf{b} \cap \mathbf{c} = \mathbf{0}$. Now, since $\mathbf{0} \neq \mathbf{0}'$, any two of the three degrees will probably be incomparable, but we are not sure how we best can prove this. (Since the three degrees are uniformly constructed it is extremely unlikely that one of them should equal $\mathbf{0}$ and one of them should equal $\mathbf{0}'$.) The construction can be generalised to yield incomparable degrees between any two degrees $\mathbf{a}, \mathbf{b}$ where $\mathbf{a} < \mathbf{b}$.

In general, we expect our degree structure to be interesting and challenging from a degree-theoretic point of view. The methods required to investigate the

structure will probably be a mixture of the number-theoretic techniques developed by Kristiansen for honest degrees and the techniques developed for degrees of computable sets, e.g. delayed diagonalisation.

7 Stronger Reducibility Relations

Our reducibility relation is slightly weak from a complexity-theoretic point of view as the number of steps in a $\mathcal{P}^-$-computation is bounded by a polynomial in the *length* of the input. (The number of steps in a $\mathcal{P}^-$-computation is bounded by a polynomial in the input x and thus by a number-theoretic expression of the form $2^{k|x|}$ where $|x|$ denotes the length of the input.) However, we believe that we can develop the structure of detour degrees for stronger reducibilities, e.g. by resorting to the function algebra $\mathcal{P}^-_b$. This function algebra is defined as $\mathcal{P}^-$ but the scheme of primitive recursion is replaced by the scheme of (primitive) recursion on notation, i.e., the scheme

$$\phi(\vec{x},0)=\xi(\vec{x}) \quad \phi(\vec{x},2y)=\psi_0(\vec{x},y,\phi(\vec{x},y)) \quad \phi(\vec{x},2y+1)=\psi_1(\vec{x},y,\phi(\vec{x},y)) \; .$$

The number of steps in a $\mathcal{P}^-_b$-computation is bounded by a polynomial in the length of the input, and $\mathcal{P}^-_{b*}$ is included in LOGSPACE.

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